Ground Return Impedance: Underground Wire with Earth Return

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SYNOPSIS: In certain transmission problems principally those relating to induction and interference phenomena, it is necessary to know the transmission characteristics of a circuit composed of an underground wire with earth return. These can be evaluated by well known engineering formulas provided the ground return impedance is known. The present paper gives the mathematical solution of this problem and shows that the ground return impedance is substantially independent of the depth of the wire below the surface.

THE object of this note is to give the solution of a problem of considerable interest and practical importance which does not appear to have been solved heretofore; this is the "ground return" impedance, per unit length, of a circuit composed of an underground wire or cable with earth return.

The physical system and the problem may be more explicitly described and explained as follows: An underground wire or cable parallel to and at depth h below the surface of the ground is surrounded by a concentric dielectric cylinder of external radius a. The earth then forms the return path for currents flowing in the wire. The ground return impedance Z_g is then defined as the ratio of the mean axial electric intensity at the external surface of the dielectric sheath to the current flowing in the wire.

When the earth extends indefinitely in all directions about the wire so that circular symmetry obtains, the problem is quite simple, and the formula for the ground return impedance, denoted in this case by $Z_g{}^0$, is well known. In practice, however, we are interested principally in the case where the wire is close to the surface of the earth, so that the distribution of return current in the ground is anything but symmetrical. For this case the formula for the ground return impedance, which it is the object of this note to state and discuss, is

$$Z_g = (1 + c)Z_g{}^0. \tag{1}$$

Here the correction term c, which takes care of the departure from circular symmetry, is given by

$$c = \frac{1}{K_0(a'i\sqrt{i})} \int_0^\infty \frac{\sqrt{\mu^2 + i} - \mu}{\sqrt{\mu^2 + i} + \mu} \cdot \frac{e^{-2h'\sqrt{\mu^2+i}}}{\sqrt{\mu^2 + i}} d\mu. \tag{2}$$

In formula (2),

$$a' = a\sqrt{\alpha},$$

$$h' = h\sqrt{\alpha},$$

$\alpha = 4\pi\lambda\omega$ where λ is the conductivity of the ground in elm. c.g.s. units, ω is 2π times the frequency,

$$i = \sqrt{-1}.$$

$K_0(a'i\sqrt{i})$ is the Bessel Function of the second kind; it is equal to $\frac{i\pi}{2} H_0^{(1)}(a'i\sqrt{i})$ where $H_0^{(1)}$ is the Hankel function as defined by Jahnke u. Emde in their Funktionentafeln. Denoted by ker a' + i kei a' the function $K_0(a'i\sqrt{i})$ has been computed and tabulated by the British Association. The only restriction on formula (2) is that the radius a is supposed small compared with the depth h.

Now the ground conductivity λ lies between 10^{-14} and 10^{-12}, while the depth h will not in practice exceed a few meters ($h \leq 10^3$). Under such circumstances, at ordinary frequencies, h' will be exceedingly small compared with unity, and a' still smaller. Consequently in evaluating the infinite integral in (2), it is permissible to take $e^{-2h'\sqrt{\mu^2+i}}$ as unity, since for $\mu > 2$, the rest of the integrand converges as $i/4\mu^3$.

Now we have

$$\int_0^\infty \frac{\sqrt{\mu^2+i}-\mu}{\sqrt{\mu^2+i}+\mu} \cdot \frac{d\mu}{\sqrt{\mu^2+i}} = \frac{1}{i}\int_0^\infty \left\{ \sqrt{\mu^2+i} - 2\mu + \frac{\mu^2}{\sqrt{\mu^2+i}} \right\} d\mu = \frac{1}{2}$$

and hence c of formula (2) becomes

$$c = \frac{1}{2}\,\frac{1}{K_0(a'i\sqrt{i})}.$$

Furthermore since a' by hypothesis is very small compared with unity, we can replace K_0 by its limiting form for vanishingly small arguments which is approximately

$$\log\,(1/a').$$

We thus get, finally, the approximate formula, valid for most practical applications,

$$Z_g = \left\{ 1 + \frac{1}{2\log\,(1/a')} \right\} Z_g^0. \qquad (3)$$

The interesting and somewhat surprising feature of this formula is that the value of the correction term $1/2 \log\,(1/a')$ likely to occur in

practical applications amounts at most to 0.05 to 0.10. On the other hand, with the wire close to the surface of the ground, the conducting area of the ground return path is just one half the area available when the ground extends indefinitely in all directions and the return impedance is Z_g^0. In other words, the departure from circular symmetry means only a very small increase in the ground return impedance. In fact this increase is so small and the ground conductivity actually so variable, the correction is hardly justified by the precision of the data, so that, in most engineering applications, we may take Z_g as equal to Z_g^0 with an error probably less than that involved in other factors, and lack of precision in data.

Derivation of Preceding Formulas

The derivation of the preceding formulas is not without interest. Since, however, this derivation is, in general, an adaptation of the methods employed in my paper 'Wave Propagation in Overhead Wires with Ground Return' (*B. S. T. J.*, Oct., 1926) it will be outlined rather than given in detail.

Take the axis of the wire as the origin and Y as the vertical axis; then the surface of the ground is the plane $y = h$. Let a unit current flow in the wire and take the axis of the wire as the Z axis. In the ground ($\rho = \sqrt{x^2 + y^2} \geqq a$) the axial electric intensity will be written

$$E = \frac{Z_g^0}{K_0(a' i\sqrt{i})} K_0(\rho' i\sqrt{i}) + E' = E^0 + E', \tag{4}$$

where $\rho' = \sqrt{\alpha}\sqrt{x^2 + y^2}$ and K_0 is the Bessel function of the second kind, related to the Hankel function by the equation

$$K_0(a' i\sqrt{i}) = \frac{\pi i}{2} H_0^{(1)}(a' i\sqrt{i}).$$

The first term on the right hand side of (4) represents the circularly symmetrical distribution which would alone exist if the surface of the ground were removed to an infinite distance, while E' is a secondary distribution due to reflection at the surface of the earth $(y = h)$. Inspection of equation (4) shows that when $\rho = a$, E is the required return impedance Z.

Strictly speaking E^0 should be written as

$$E^0 = \frac{Z_g^0}{K_0(a' i\sqrt{i})} \{K_0(\rho' i\sqrt{i}) + h_1 K_1(\rho' i\sqrt{i}) \cos\theta + h_2 K_2(\rho' i\sqrt{i}) \cos 2\theta + \cdots\}, \tag{5}$$

the harmonic terms representing secondary reflection at the surface of the dielectric cylinder ($\rho = a$). If a is made sufficiently small, however, the harmonic terms become negligible. In view of this fact, the large amount of tedious additional analysis required, if the harmonic terms are retained, is not believed to be justified by the practical applications contemplated. E^0 will therefore be taken as in formula (4).

The secondary electric intensity E' can always be written as the Fourier integral

$$E' = \int_0^\infty F(\mu) e^{y'\sqrt{\mu^2+i}} \cos x'\mu \, d\mu, \quad 0 \leqq y \leqq h, \quad (6)$$

where $x' = x\sqrt{\alpha}$, $y' = y\sqrt{\alpha}$, and the Fourier function F is to be determined. For the formulation of the boundary conditions at $y = h$ we also require the expansion of $K_0(\rho' i\sqrt{i})$ as a Fourier integral; the required expansion is *

$$K_0(\rho' i\sqrt{i}) = \int_0^\infty \frac{1}{\sqrt{\mu^2 + i}} e^{-y'\sqrt{\mu^2+i}} \cos x'\mu \, d\mu, \quad \rho > 0. \quad (7)$$

In the dielectric, the magnetic forces H_x, H_y are taken as

$$\left.\begin{aligned} H_x &= \int_0^\infty \phi(\mu) e^{-y\mu} \cos x\mu \, d\mu \\ H_y &= -\int_0^\infty \phi(\mu) e^{-y\mu} \sin x\mu \, d\mu \end{aligned}\right\} \quad y \geqq h. \quad (8)$$

In the ground, on the other hand, we have

$$\begin{aligned} i\omega H_x &= -\frac{\partial}{\partial y} E, \\ i\omega H_y &= \frac{\partial}{\partial x} E. \end{aligned} \quad (9)$$

In order to satisfy the boundary conditions at the plane $y = h$, we equate H_x as given by (8) and (9), and H_y as given by (8) and (9). The explicit formulas for H_x and H_y are derived from (9) by substituting the Fourier integral for K_0, as given by (7), in (4) and differentiating as indicated.

The two equations resulting from equating the two expressions for H_x and the two expressions for H_y can be solved for the Fourier function

* See note at end of this paper.

$F(\mu)$. With this determined the required impedance Z_g is simply by (4)

$$Z_g = Z_g^0 + E_\rho' = 0 \tag{10}$$

on the assumption that $a' = a\sqrt{\alpha}$ is quite small compared to unity. This gives formula (1).

Note: The expansion (7), which is believed to be novel, was derived by a limiting process rather too long and unsuitable for inclusion in this paper. It and the following additional expansions are quite useful in certain problems on wave propagation.

$$\cos\theta\cdot K_1(\rho' i\sqrt{i}) = -\sqrt{i}\int_0^\infty e^{-y'\sqrt{\mu^2+i}}\cos x'\mu\, d\mu,$$

$$\sin\theta\cdot K_1(\rho' i\sqrt{i}) = -\sqrt{i}\int_0^\infty \frac{\mu}{\sqrt{\mu^2+i}} e^{-y'\sqrt{\mu^2+i}}\sin x'\mu\, d\mu,$$

where

$$\cos\theta = y/\rho = y/\sqrt{x^2+y^2},$$

$$\sin\theta = x/\rho = x/\sqrt{x^2+y^2}.$$